A neural model of multidigit numerical representation and comparison

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A Neural Model of Multidigit Numerical Representation and Comparison

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Abstract

The Extended Spatial Number Network (ESpaN) is a neural model that simulates processing of high-level numerical stimuli such as multi-digit numbers. The ESpaN model targets the explanation of human psychophysical data, such as error rates and reaction times, about multi-digit (base 10) numerical stimuli, and describes how such a competence can develop through learning. The model suggests how the brain represents and processes an open-ended set of numbers and their regularities, such as the place-value structure, with finite resources in the brain. The model does that by showing how a multi-digit spatial number map forms through interactions with learned semantic categories that symbolize separate digits, as well as place markers like "tens," "hundreds," "thousands," etc. When number-stimuli are presented to the network, they trigger learning of associations between specific semantic categories and corresponding spatial locations of the spatial number map that together build a multi-digit spatial representation. Training of the network is aimed at portraying the process of development of human numerical competence during the first years of a child's life. The earlier SpaN model proposed a spatial number map, which both human and animal possess in their Where cortical processing stream, that can explain many data about analog numerical representation and comparison. The ESpaN model shows how learned cognitive categories in the What cortical processing stream can extend numerical competence to multi-digit numbers with a place-value structure. The ESpaN model hereby suggests how cortical cognitive and spatial processes can utilize a learned What-and-Where interstream interaction to control the development of multi-digit numerical abilities.

Introduction

The mighty chariotry, twice ten thousand, thousands upon thousands,
the Lord came from Sinai into the holy place.
Psalm 68

Numbers and mathematics, the science of numbers, are commonly associated with digits and mathematical operators that rely heavily on symbolic notation. Historically, however, it was not written symbolic notation, but spoken language that brought the concept of number into human life. Long before the appearance of the first concise numerical notation, in early Sumerian language dating from the third millennium BC, number-words reflected the structure of the numerical system, as shown in Figure 1 (after Menninger, 1969). Numerical systems were developed independently by many civilizations in different parts of the world. The Sumerians, who inhabited the southern part of Mesopotamia, based their system on gradations of the number 60, an influence that can be seen today in how time is measured in minutes and seconds (Figure 1, left). The Celts in Europe as well as the Maya and the Aztecs in Mesoamerica used a vigesimal, or base-20, numerical system. Modern French still bears the legacy of the base-20 that interferes with its number-naming base-10 structure. Our modern numerical competence has a decimal system in its foundation that originated from Arab and Indian cultures.

Sumerian

Value	Cuneiform symbol	Number word
1		*aš*
10		*u*
60		*geš*
60 10		*geš-u*
60^2		*šar*
60^2 10		*šar-u*
60^3		*šar-gal*

Egyptian

Value	Symbol
1	I
10	
100	
1000	
10000	
2374	

Roman

Value	Symbol
1	I
10	X
50	L
100	C
500	D
1000	M
2374	MMCCCLXXIIII

Figure 1. Sumerian, Egyptian, and Roman number systems. The Sumerian language illustrates how the structure of a number system was reflected in their number-words. Egyptian and Roman languages provide examples of number-systems formed according to an additive principle.

Initially, most of the number systems were based on an *additive* principle. Egyptian and Roman systems (Figure 1, middle, right) serve as good examples of how the symbols for units or hundreds are ordered and then grouped together such that their sum represents a new symbol for ten or thousand, respectively. The additive principle allowed use of a compressed representation of large numbers, such as 2,374, but this representation was not as compressed and convenient for calculations as the modern number system based on a *multiplicative* principle. In a number system based on a multiplicative principle, maximum compression is achieved by means of

place-values. Instead of having a new symbol for each of the powers of ten, as in the case of Egyptian hieroglyphs, the power of ten is encoded by its place information. Such a system was used by Babylonians as early as about 2000 BC, with only one principal difference from the modern number system: they were lacking the concept of zero. In Babylonian notation, a number such as 3,005 could not be expressed unambiguously, as the empty space was used instead of zeros. This limitation was the source of possible confusion and slowed down the development of mathematics.

Despite their many differences, the vast majority of number systems had a few features in common: they utilized some form of compressed representation of an open-ended set of numbers by means of either an additive or multiplicative principle. Other similarities include the fact that the number-names employed as categories for the compressed representation often reflected the structure of the number system, such as in case of Sumerian and Roman systems. More importantly, even if the symbolic notation was based on an additive principle, the linguistic structure relied on a multiplicative relation. This is especially surprising to find with the Romans, who had a precisely ordered flexible verbal number sequence (Figure 1, right), but used a rather crude and cumbersome symbolic notation. The similarities of the number systems that appeared in different parts of the world are perhaps not surprising, since their development was shaped by the common needs to describe the environment and to communicate it within a community.

An analysis of the historical development of numerical competence leads to the following conclusion: a natural task that produces common abstract concepts and common linguistic representations may suggest a common representation in the brain. It can be assumed that this representation arises from a more basic representation of numerical quantities, one that builds on an internal spatial representation in the brain for numerical estimation and comparison (Dehaene, 1997; Grossberg and Repin, 2000). The present article models how this spatial numerical representation can be extended into a multi-digit numerical representation through its learned interactions with number category names.

Three types of models for multi-digit number comparison

Psychophysical data related to multi-digit number processing include studies on number reading, comparison, and simple arithmetic. A major controversy arising from the data concerns the response times in numerical comparison experiments (Hinrichs et al., 1981; Poltrock and Schwartz, 1984; Dehaene et al., 1990; Brysbaert, 1995). All results agree on the general trend when the numbers are compared to a fixed standard: the response time becomes longer as the difference between the presented number and the standard becomes smaller. This trend reflects the temporal side of the Numerical Distance effect (Dehaene, 1997). In addition to these response time differences, the Numerical Distance effect is exhibited in an increasing error rate as the difference between the number being compared decreases. The controversial portion of the data is related to how the reaction times behave at a decade boundary (for two-digit numbers). Experiments on two-digit (Dehaene et al., 1990) and multi-digit (Poltrock and Schwartz, 1984) number comparisons reported no fine-grain patterns in the reaction time data beyond the conventional Numerical Distance effect. In contrast, the study by Hinrichs et al. (1981) mentioned a statistically significant increase in the reaction time change for the two boundaries between the decades (49-50 and 59-60) versus the adjacent intervals in the number comparison experiment with stimuli ranging from 11 to 99 and a fixed standard of 55. The experiments by Bryzbaert (1995) demonstrated a reverse distance effect (reaction time increase for larger numerical distance) for the two-digit numbers across the decades boundaries (Figure 2). Neither

Hinrichs et al. (1981) nor Bryzbaert (1995) proposed a mechanism to explain these observed paradoxical results for numerical comparison between decades or on decades boundaries.

Traditionally, explanations of psychophysical data about multi-digit number comparison were based on the information about the symbolic and linguistic structure of numbers – called the *lexicographic* approach – or the magnitude representation of numbers – called the *holistic* approach – or a combination of both. The lexicographic approach predicts numerical comparison times based solely on the leftmost digit information (the decades digit in the case of two-digit numbers), completely ignoring the other information (Poltrock and Schwartz, 1984). The holistic approach – in which the symbolic numerical notation first would be converted to a magnitude representation and only then the would two numbers be compared – is supported by the experiments by Dehaene et al. (1990). A combination of the two approaches above was proposed in Hinrichs et al. (1981). According to their hypothesis, for two-digit numbers, the result of the units comparison could influence the result of the decades comparison that, by itself, was providing the correct result. The Hinrichs et al. (1981) hypothesis, called the *interference* model, was questioned by Dehaene et al. (1990), based on the results of their experiments with asynchronous presentation of decades and units digits during the two-digit number comparison task. These experiments yielded no difference in the error rates and reaction times for the conditions when either decades or units digits were presented 50 ms earlier than the other digit. In ruling out the interference model in favor of the holistic model, a strong emphasis was placed on the relative processing speed of the units and decades digits. According to the Dehaene et al. (1990) argument, the earlier presentation of the units digit should have increased the reaction time, while the earlier presentation of the decades digit should have reduced the reaction time, results which have not been observed in the experiments.

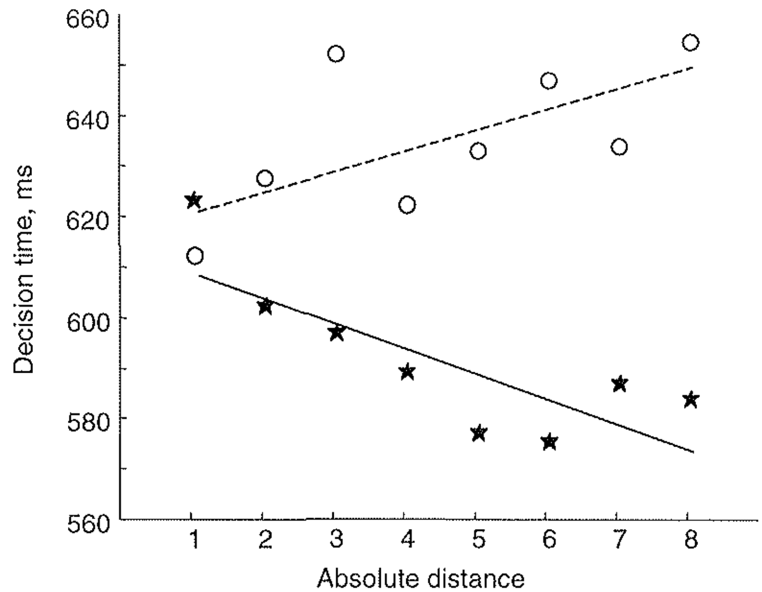

Figure 2. Decision times for two-digit number comparison as a function of the absolute distance between the target and comparison numbers, whether or not the target and the standard shared the same decade. Open circle: different decade; solid star: same decade. [Adapted with permission from Bryzbaert (1995).].

As noted above, the present work develops a model of cognitive numerical representation in the human brain that incorporates both lexicographic and holistic components. The lexicographic mechanisms help to account for the complex structure of the modern numerical system, including the place-value principle that allows a compressed representation of the open-ended set of numbers. In particular, this aspect of the model shows how learned number-name categories are involved in numerical representation, and thus clarifies how cognitive processes begin to enter the symbolic number system. The holistic approach provides a basis for the spatial representation of numerical information in the brain and fundamental mechanisms underlying the number comparison processes. Why does not a spatial representation alone have the capacity to

represent arbitrarily large finite quantities? One reason is that an extended liner array of spatially represented numbers would run out of space in the brain. A deeper reason is that such a linear array exhibits a Weber law property (c.f. Figure 5, panel I, below), wherein larger numbers have a coarser resolution, thereby leading to increasingly inaccurate operations with them.

In reality, people can deal with arbitrarily large numbers without losing much accuracy. Therefore, additional means for creating an adequate representation for numerical information are required. We suggest herein how the higher-level cognitive process of categorical perception is closely related to the symbolic structure of any number system. Our model posits learned interactions between number categories (that are themselves learned within the brain's What processing stream for language acquisition) and the representation of numbers in a spatial map (that is part of the brain's Where processing stream for spatial representation and action). The model hereby predicts that symbolic numerical abilities arise through a What-Where interstream interaction. It is through the learned interactions between cognitive number categories and the primal spatial number representation that the model can show why and how the open-ended nature of numerical representation arises.

For example, in the English language, number units, tens, hundreds, thousands, etc. are used, with each successive group of numbers tending to repeat the basic number units, but with an order of magnitude more numbers represented in each group. How is this numerical proliferation represented in the brain? Our neural model, the Extended Spatial Number Network, or ESpaN, an extended version of the SpaN model of Grossberg and Repin (2000), proposes how the brain combines both the linguistic categories that denote number-names and the spatial substrate of the basic numerical representation in a single computational framework.

The ESpaN model provides a quantitative fit to both the error rates and the reaction time data for multi-digit numbers. It simulates the reaction times and suggests the explanation of the paradoxical reversed Numerical Distance effect observed in Bryzbaert (1995) and partly indicated in Hinrichs et al. (1981). The ESpaN also simulates the numerical comparison results for the two-digit numbers for asynchronous digit presentation paradigm and points out the difference in two-digit number comparisons for different language structures, such as English (24 is pronounced as *twenty-four*) versus Dutch (24 is pronounced as *four-and-twenty*). The scope of the model is restricted to humans, because as far as we know, animals do not have names for categories. The next section describes the structure and equations of the ESpaN model, focusing on the interaction between number-name categories and the spatial number map through learning. The model is then used to simulate the reaction time and error rate data in a multi-digit number comparison task as well as the example with asynchronous digit presentation. Finally, we discuss the evolutionary implications of the proposed model, and its limitations.

The ESpaN model

The essence of the ESpaN model is the fusion of verbal categories for number-words and spatial analog numerical representation, which may be considered as a fusion of What and Where information streams (Figure 3). Recent neurophysiological data have begun to demonstrate the existence of such What-and-Where interaction in the primate brain (Rainer et al., 1998). In the present example, the Where stream is represented by the spatial number map that has a specific topological structure which has a brain correlate in the inferior parietal cortex (Dehaene et al., 1996; Pinel et al., 1999). This spatial number map may be activated through the sensory input coming from visual, auditory or other modality, as well as by the cognitive categorical inputs originating in other cortical areas. The What stream is represented by a set of

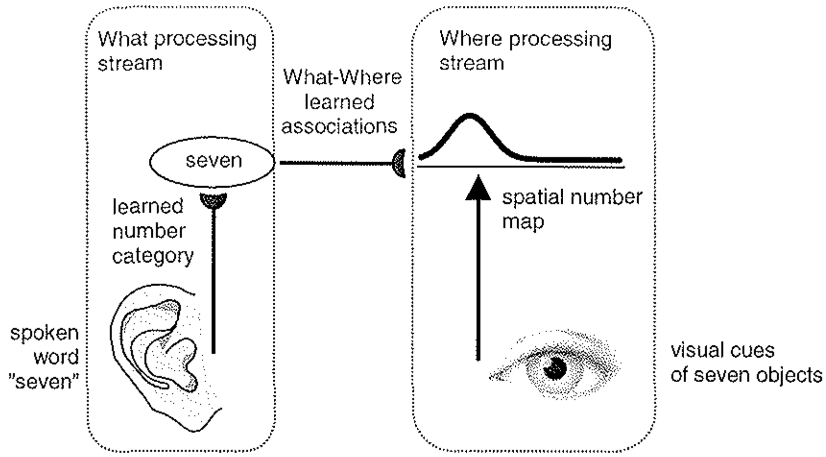

Figure 3. What-Where cortical processing stream fusion is a key hypothesis of the ESpaN model. Previously learned phonetic categories in the What stream become associated with corresponding locations of the spatial number map in the Where stream. These learned What-Where associations are essential for building a number-system based on the place- (Where) value (What) principle.

verbal categories that are learned from speech representations in the prefrontal and temporal cortical areas that have projections from the auditory cortex (Gruber et al., 2000; Grabowski et al., 1998). The verbal categories are connected to the spatial map by adaptive memory weights via associative learning, and are activated by phonetic number-names.

A two-dimensional spatial map serves as the basic structure for multi-digit number representation, and the representation of each particular number is accessed through the activation of a corresponding category. For example, *two hundred* requires categories *two* and *hundred* in order to activate the corresponding region in the spatial number map. Before What-Where learned associations form, the basic spatial number map has a one-dimensional structure that is represented by extended strips across the map (Figure 4A). What-Where learning converts these strips into localized regions that represent multi-digit numbers (Figure 4B). Using this learned representation, the ESpaN model can explain numerical abilities for the number

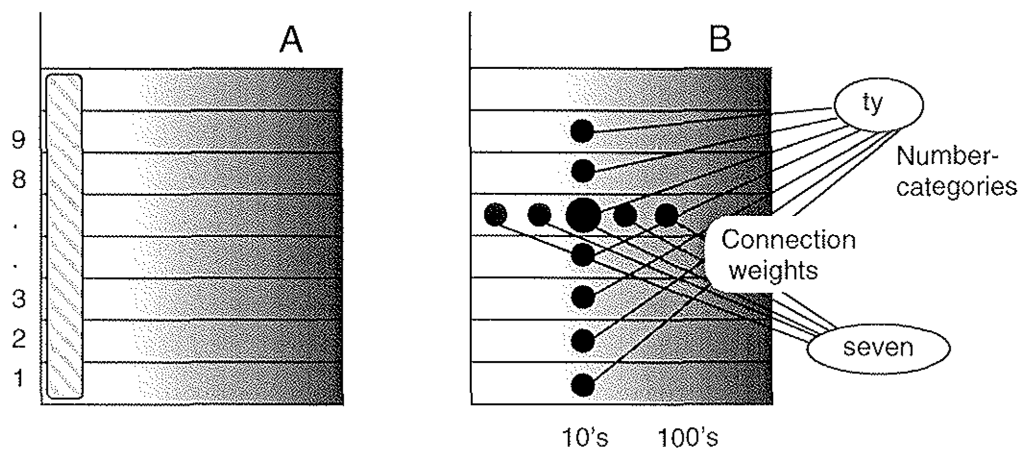

Figure 4. Schematic representation of the spatial number map and learned What-Where associations. A: The striped area on the left shows the location of the primary (units) weights strip. B: An example of where the association for *seven-ty* is formed in the spatial map. The size of the solid circles encodes weight magnitude; the strongest association for *seventy* is arises at the spatial location where both the associations for categories *seven* and *ty* are present.

system based on both additive and multiplicative principles, as long as there is a structure of linguistic categories for number names. In order to demonstrate and test the ESpaN approach, we

have developed a computational model simulating the modern number system based on English number naming and the decimal place-value number system.

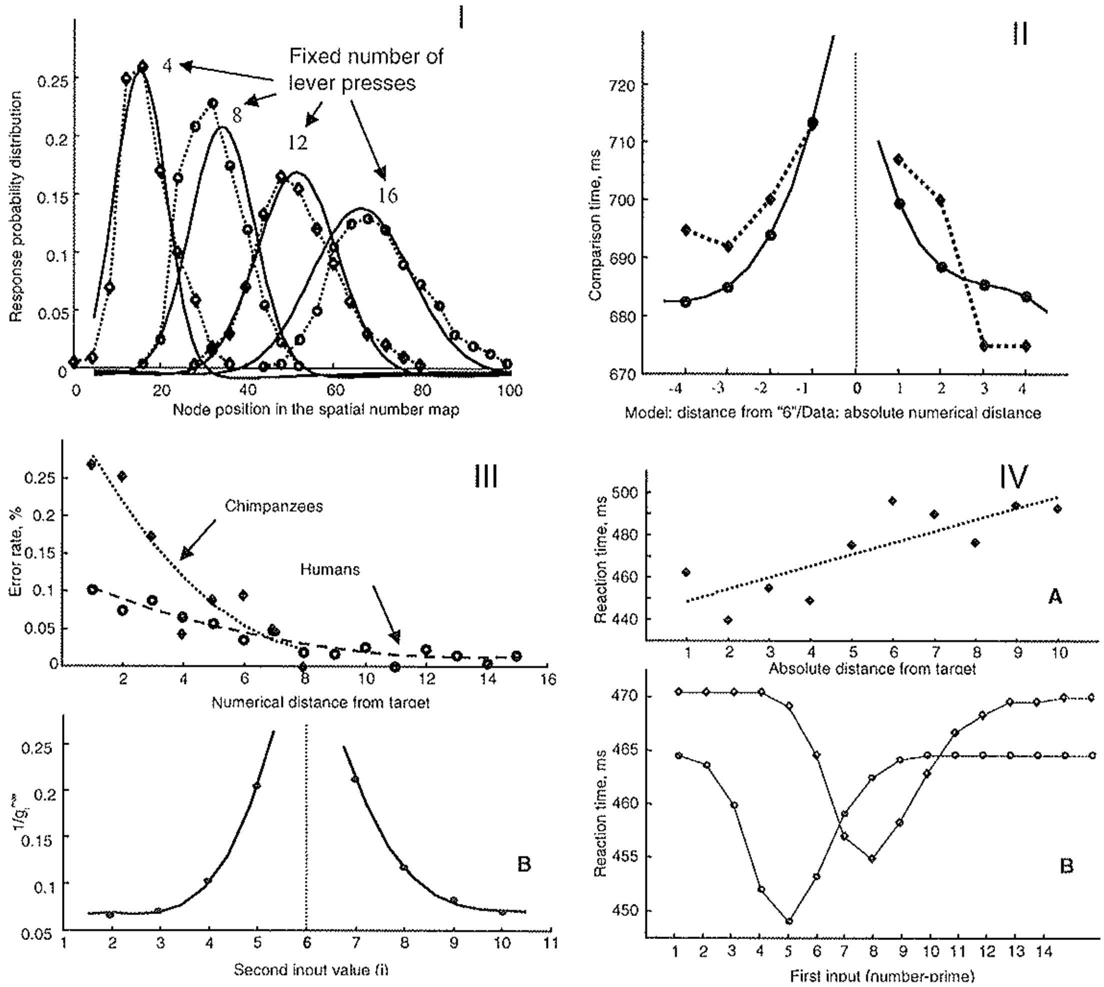

Figure 5. Summary of SpaN model simulations. I: Response distribution. Data: Rats were trained to press a lever a fixed number of times before switching to another lever in order to get a reward. Data points (diamonds, circles) represent the proportion of responses made to match one of the four required fixed numbers of lever presses (4, 8, 12, or 16). [Adapted with permission from Mechner (1958)]. Model: Solid lines show equilibrium activities $p_i(\infty)$ of the spatial number map for four inputs corresponding to 4, 8, 12, and 16 events. II: Number comparison times as a function of numerical distance. Data: diamonds, dashed lines. People compared two-digit numbers presented in visual Arabic notation. [Adapted with permission from Link (1990)]. Model: circles, solid lines. III: A: Error rates for chimpanzees selecting the larger pile of chocolate bits [Adapted with permission from Washburn and Rambaugh (1991)] and people comparing two-digit numbers to a fixed standard of 65 as a function of the numerical distance from the target [Adapted with permission from Dehaene et al., (1990)]. B: Inverse of the maximum value of the winning comparison wave (see Equations (12) and (13) of this article) as a function of distance between the first and the second inputs. IV: Number priming. A: Best linear fit (dashed line) to the data of Bryzbaert (1995). Adapted with permission. B: Model simulations of the priming effect for number targets 5 (circles) and 8 (diamonds).

Number categories and spatial organization

The original SpaN model simulated an ordered analog one-dimensional spatial map, where activations of the smaller numbers were positioned towards the left side of the map and those of the larger numbers, towards the right side. The *left* and the *right* sides of the map reflected the commonly observed property of ordering of the small to large quantities in the direction from left to right. All the simulations of numerical abilities in the SpaN model relied on how sequences of events activated the map or caused a redistribution of activation along its one-dimensional array. Figure 5 summarizes the data simulations of the SpaN model.

The ESpaN model proposes a natural extension of the spatial map hypothesis. It suggests that the activation of the spatial number map is not confined to the narrow strip of a number line, but rather decreases gradually in the dimension approximately orthogonal to the number line. Thus it is more accurate to say that number *strips* are activated in the spatial number map. Without loss of generality, it is assumed that the activation decreases monotonically as described by a simple gradient (Equation (4) below).

The ESpaN model suggests how neural connections from number-categories to the nodes that form the two-dimensional spatial number map are tuned through learning. These number-categories represent the phonetic entities that reflect the linguistic structure of the particular number system. In English, they are the single digits from *one* to *nine*, group categories such as *hundred* or *thousand*, and specific phonetic markers such as *ty*, that denotes tens in *twenty* or *thirty*. These number-categories may be more complex, such as in French or Basque, reflecting the mixture of base-10 and base-20 systems. Phonetic number-categories may also exactly reflect

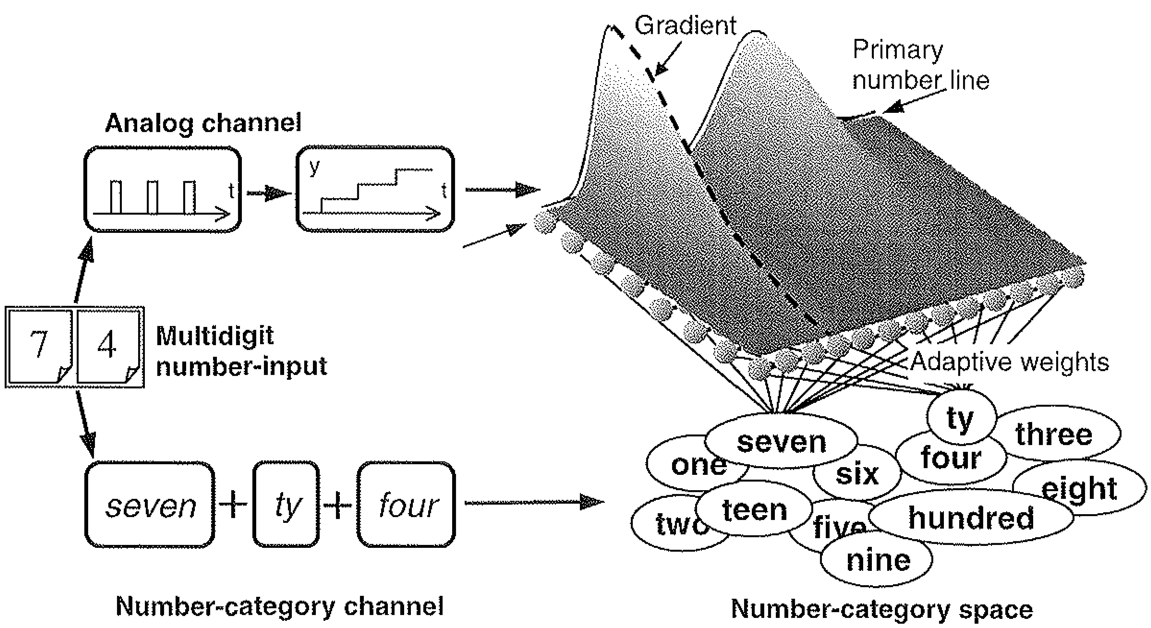

Figure 6. ESpaN model in the learning mode. During the initial developmental, the analog input channel provides the necessary numerical input that drives the formation of the primary strip in the spatial number map. Later in the development, with the acquisition of language, number-words reflecting learned number-categories provide the input for the other channel: The activation of the primary strip due to the analog channel propagates down the gradient across the spatial map. When both input channels are active, the weights are learned in the locations of the spatial map activated simultaneously by both the number-category input and the gradient of activation.

the decimal structure of the number system with the exception of *eleven* and *twelve* in the teens and awkward structures such as *twen-ty* instead of *two-ten* as in modern Chinese.

The schematic operation of the model during the learning phase is shown in Figure 6. Numerical inputs may activate both the analog and the verbal input channels. The analog channel represents such processes as counting and estimation of quantities that rely on a spatial representation. In the original SpaN model, this input channel was implemented by means of a preprocessor module that converted the number of items in a spatial pattern or the number of events in a temporal sequence into an analog amplitude representation which activated different spatial locations in the number line. The verbal channel represents number-categories that correspond to the number-names. Number input *seventy-four* is assumed to activate such categories as *seven*, *ty* (for the tens), and *four*.

It is known that human infants who are only a few months old are capable of distinguishing small quantities (Wynn, 1998), suggesting early development of a basic spatial number map. With the acquisition of language, the number-words that denote the small single-digit numbers begin to influence the number representation process. During learning within the model, these learned number-categories are associated with the area of the spatial number map that has the highest activation level at the time when the number-category input is active. For one-digit numbers, this area represents the original, or the primary, number line. Later in map development, when learning more complex two- or three-digit numbers, new categories such as *teen*, *ty*, and *hundred* are learned and become associated with areas of the number map that have smaller activation than the primary number line. Learning of categories *ty* or *hundred* occurs in the presence of a single-digit category; for example, *seven* for *seven-ty* or *seven hundred*. This means that weights for *ty* or *hundred* in the case of *seven-ty* or *seven hundred* will be learned within that portion of the map gradient, where *seven* is active (Figure 4). If a gradual exposure to more complex numerical structures is assumed, and tens tend to be learned before hundreds, then the categories corresponding to tens will be learned at approximately the same values of the gradient orthogonal to the primary number line. Thus, the structure of the spatial number map after learning both one- and two-digit numbers will represent a strip of a primary number line corresponding to the learned one-digit numbers and another strip, somewhat parallel to it, corresponding to the tens, or the *ty*, category. Such a strip structure of a spatial number map may not be very regular, but it tends to have a topology where one dimension represents the analog quantitative scale and the other dimension spans a certain number of categories that expand the numerical system with progressively larger number of digits.

Figure 7 presents an example of the dynamics of the simulated two-dimensional weight pattern connecting number-categories with the spatial number map during the learning phase. To roughly portray the first steps of how the child may learn numbers, only one-digit numbers were presented to the network at first. Figure 7A (top row) shows how the weights for number-categories from one to nine evolve with presentation of more and more inputs. The growing strip of large weights connect single-digit categories to the locations of the spatial number map along the primary number line. This strip will be referred as the *units strip* later on. As with a child learning numbers from simple to more complex, the two-digit numbers were then added to the training set of the ESpaN network.

Figure 7A (bottom row) shows that once the category weights near the primary number line have saturated, weights corresponding to the new inputs, the tens, are learned within the strip that is parallel to the primary number line and located to the right of the units strip. This *tens strip* represents how the weights from single-digit categories from 1 to 9 are associated with the decades digit of the two-digit numbers.

In our simulation example, we used the English language, which does not possess the most optimal structure of number naming, and bears odd artifacts of the past such as *eleven* and *twelve*, and to a lesser extent, the whole structure of *teens*. One may argue that numbers 11 and 12 fall out of the teen linguistic structure. In this case, these two numbers may be assigned individual categories *eleven* and *twelve* that would be learned further to the right from the basic numbers (1 through 9) on the primary number line. The latter case does not contradict the model hypothesis, as we know that primary number line does not have to end at 9, but may extend as high as 50 for some animal species; see Grossberg and Repin (2000) for further discussion.

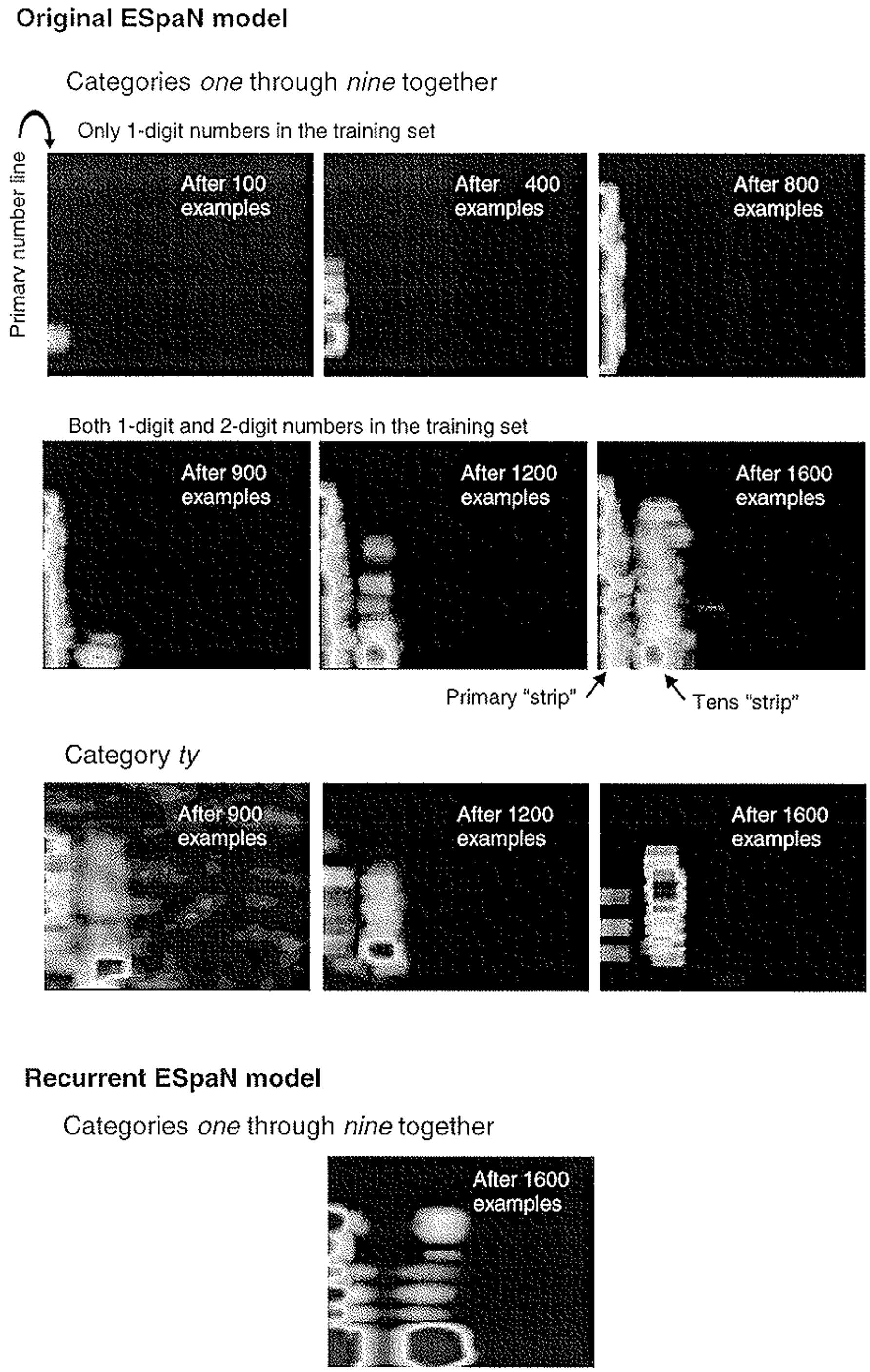

Figure 7. Computer simulation of learned weight amplitudes for different categories: magnitude is coded by the shades of gray from dark (small) to light (large). A: learning progress for weights connecting categories *one* through *nine* to the spatial number map; weight patterns for categories *one* through *nine* are plotted together on each panel. B: learning progress for only the weights connecting category *ty* to the spatial map; no learning for category *ty* occurs when only single-digit numbers are present in the training set. C: learned weight pattern for recurrent ESpaN formulation for categories *one* through *nine* plotted together; this weight pattern represents the same stage of network learning as the bottom right panel of part A of this figure.

The formation of the strip structure of the weights to the spatial number map is clarified in Figure 8 in the example of how the weights are learned from each category. It illustrates the change of the weights connecting the number-category *five* to the spatial number map. When

only one-digit numbers are presented, the weight patterns with each number line resemble the activation produced by the analog representation of the input corresponding to five items or events as a result of processing by basic mechanisms of the SpaN model (Figure 8, top three panels). Incorporation of two-digit numbers into the learning process leads to formation of the part of the tens strip that corresponds to the number-category *five* (Figure 8, bottom three panels). The structure of the weight pattern deviates from the regular bell-shaped activations obtained from in the SpaN model due to competitive interstrip interactions during the learning process and the stochastic nature of weight initialization and ordering of the training set.

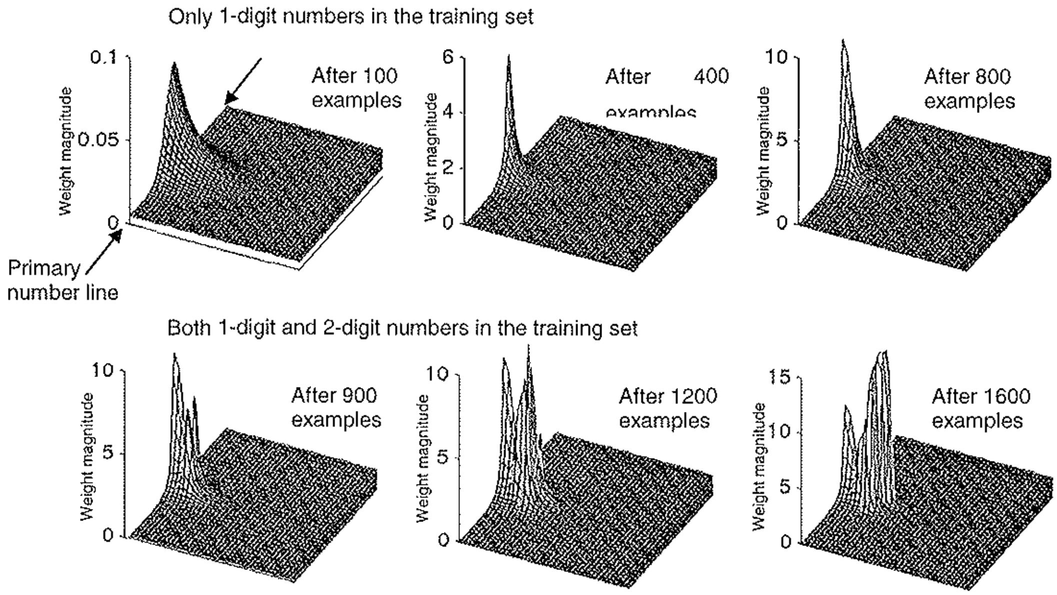

Figure 8. Computer simulation of the learning process for category *five*. When only 1-digit numbers are present in the training set (top three panels), weights in the primary strip are learned. Presence of 2-digit numbers results in weights growing further from the primary number line, in the *ty* strip (bottom three panels). The irregularities of the weight patterns observed in the figure are due to the initialization of weights to small random numbers and random order of training examples in the training set.

In addition to the weights from single-digit categories, the weights connecting *ty* (or the tens) category to the spatial number map are also learned. No learning for the *ty* category occurs when only the single-digit numbers are presented. When the two-digit numbers are included in the training set, a strip of weights spanning the numbers from 2 to 9 along the primary number line dimension is learned in approximately the same location as the strip that corresponds to the category *ty* (Figure 7B). Note that numbers from 11 to 19 do not contribute to the learning of the tens strip, since in English they are formed by a separate phonetic structure such as *teen*, as in *four-teen*, and represent a separate category. Model equations for the map learning are given at the end of the article.

Multi-digit number comparison

The second part of the ESpaN model embodies a mechanism by which the proposed spatial number map gets incorporated into simple operations with numbers. This mechanism represents an extension of the comparison wave mechanism proposed in the original SpaN model

(Grossberg and Repin, 2000). The original comparison wave was able to model successfully such properties of human and animal numerical comparison as reaction times and error rates.

The comparison process occurs when two number-inputs are presented to the network with a zero or short delay between them. The activations of different numbers peak at different locations within the spatial number map. Build-up of activation due to a second input starts while the activation of the first input is still present but may be decaying. The sum of this correlated, but spatially displaced, growth and decay of activation produces a bell-shaped activation whose peak moves continuously from the location of the first input to that of the second input. Such a moving bell-shaped activation was called a *comparison wave* in Grossberg and Repin (2000). In neural models of motion perception, such waves have successfully simulated many data about long-range apparent motion (Grossberg and Rudd, 1989; 1992; Baloch and Grossberg, 1997; Grossberg 1999). This fact illustrates our hypothesis that many properties of numerical estimation have arisen from properties of spatial representation and motion processing in the Where cortical processing stream; see Grossberg and Repin (2000) for further discussion. In the ESpaN model, multiple comparison waves exist to represent the redistribution of activation patterns across the two-dimensional spatial number map in a direction *parallel* to the primary number line. In other words, if one looks at the spatial number map as a set of number lines that are parallel to the primary number line, then multiple comparison waves occur within the

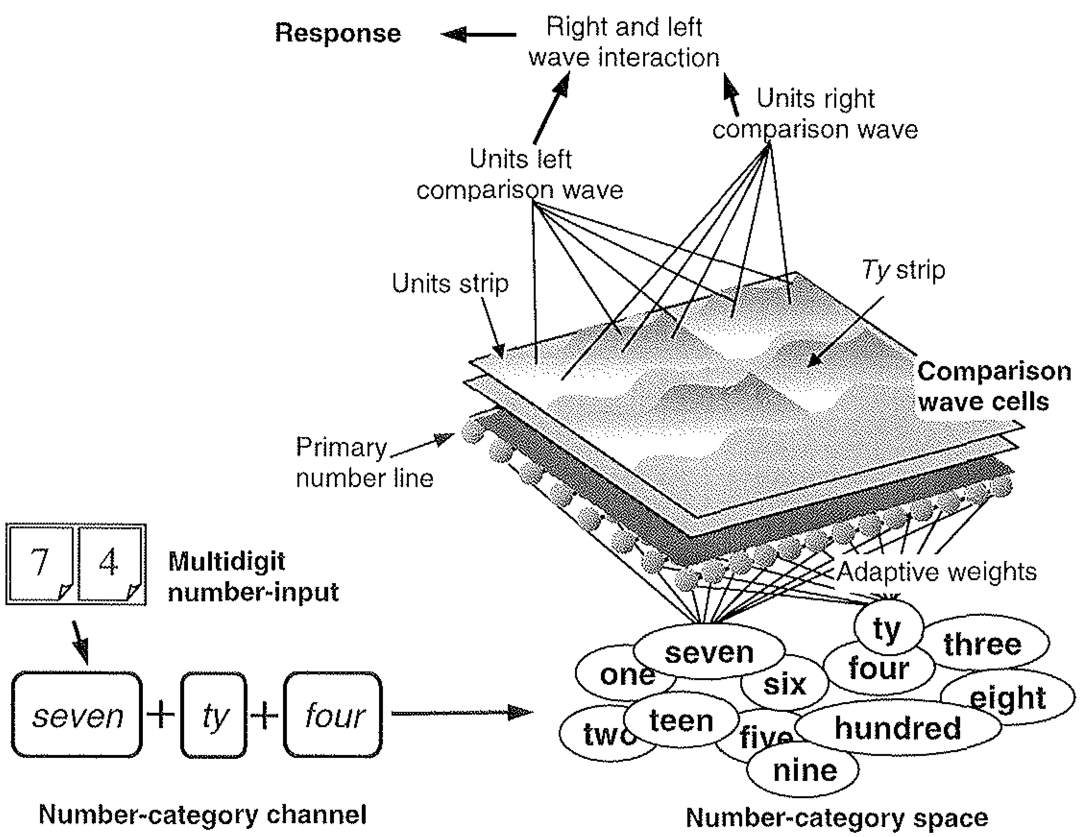

Figure 9. ESpaN model in the comparison mode. Multi-digit numerical input activates categories in the number-category space. These categories activate specific regions within corresponding strips of the spatial number map via connection weights formed during learning. Dynamic redistribution of activation across the spatial number map is detected with the help of the comparison wave cell layers. The direction of comparison (right or left) is determined from the interaction of comparison waves that occur in different strips.

individual number lines. The ensemble of all these waves determines judgements in the manner described below.

The operation of the ESpaN model in the comparison mode is described by the diagram in Figure 9. Here, the number comparison process is assumed to take place after a sufficient amount of learning has been accomplished, and a weight pattern from number-categories to the two-dimensional spatial number map (similar to the example in Figure 7) has been formed. The learning stage of the ESpaN model (and the original SpaN model) assumes that the signal whose amplitude is proportional to the numerical input is generated by the analog input channel. With the weight pattern connecting number-categories to the spatial map already in place, the input from the analog channel does not have to be present. As children learn more, they become less dependent on the primitive process of counting on fingers. They rely more on numbers that are expressed in their symbolic form with the help of spoken number-names or written number-symbols. Similarly, the ESpaN model assumes that category-based input channel takes a major role in number processing after learning has been completed. It is then sufficient for the number-inputs to directly activate the number-categories that project through the weights onto the spatial number map, producing activation patterns similar to the ones that would have been produced through the analog input channel alone. During the learning stage, the category input reflects the phonetics of the number-naming structure of the language. As in the case of development of mathematical skills, when children start to operate without difficulty with both multiplication tables learned in verbal format and Arabic number-symbols, the model assumes that numerical information in both visual and auditory modalities can activate the category input channel and thus the corresponding number maps and comparison waves.

Simulated comparison waves for number pairs (32,55) and (38,55) are shown in Figure 10. In both examples, the processing of the decades digit starts before the units digit as reflected in the input temporal structure. Separate comparison waves occur within the tens and units strips of the spatial number map. In case of 32 and 55, waves to the right in the tens strip (3<5) and in

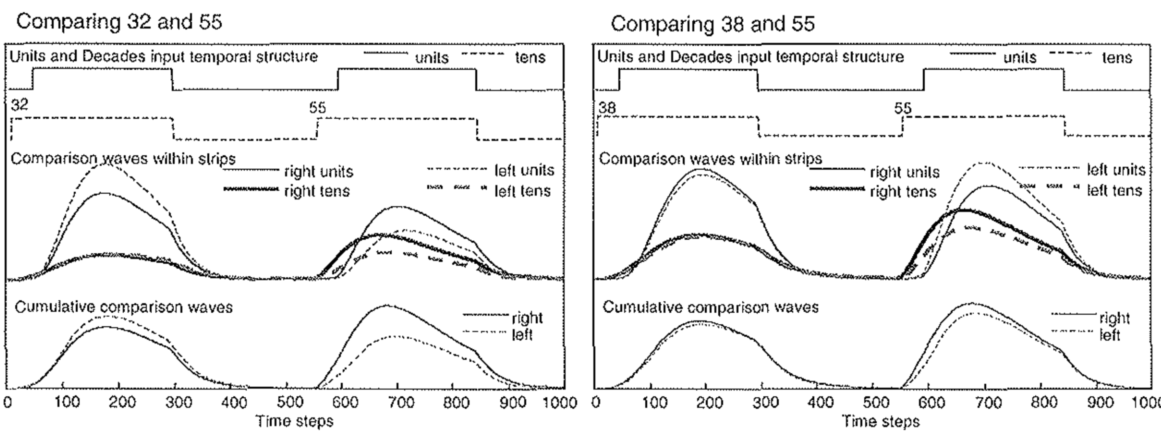

Figure 10. Simulation of the comparison wave for number pairs 32<55 (left) and 38<55 (right) presented in the numerical comparison task. Two top lines on each panel illustrate the time course of a two-digit number presentation: units digit follows the decades digit after a short delay. Middle graph shows that comparison waves occur in both tens and units strips in both directions. Note that only the comparison wave that occurs after the onset of the second input contributes to the response. The comparison wave before 500 time steps is a by-product of the growth and partial decay of the activation of the first input. The inter-strip interaction results in a cumulative left and right comparison wave (bottom graph). The comparison wave with a larger magnitude wins and determines the response: if the larger wave to the right, then the second number is larger; if the larger wave is to the left, then the second number is smaller.

the units strip (2<5) are larger than waves to the left. When these waves are added up, the cumulative comparison wave moving to the right is significantly larger than the one moving to the left (bottom portion of Figure 10). This result corresponds to the judgement that 55 is larger than 32. Such an interaction between the comparison waves is assumed to occur during the output processing stage or shortly before the response is produced. In case of 38 and 55, the right wave in the tens strip (3<5) is again larger than the left one, but the opposite situation occurs in the units strip (8>5), where the wave moving to the left wins. The cumulative right comparison wave is still larger for this pair of inputs because of the assumption that attention more heavily weight the tens strip; see Equations (10) and (11) below, where the weighting coefficients modulate the level of attention. The difference between the left and the right is, however, smaller than for the number-pair 32 and 55, thereby leading to more errors and slower RTs. Model equations for the comparison wave process are given at the end of the article.

Data simulations

The ESpaN model allows simulation of the two-digit number comparison experiments and thereby offers an explanation of reaction times data (Bryzbaert, 1995) about the reversed distance effect. A decades-units interaction mechanism based on the cumulative properties of multiple comparison waves is proposed to underlie this paradoxical effect. In the simulated paradigm, a pair of two-digit numerical stimuli were presented to the subject with a stimulus onset asynchrony (SOA) between the first and the second ranging from zero to several seconds. Even for the zero SOA, serial processing of the stimuli assures that processing of the second number starts after the first number has already begun to be processed. Processing of a composite stimulus such as a two-digit number was treated as a mixture of parallel and serial mechanisms in the following fashion: the input signal corresponding to the tens digit started a few milliseconds before the input signal corresponding to the units digit. After this brief delay (denoted UTA, Units-Tens Asynchrony), both tens and units inputs were present simultaneously.

The simulations were implemented in MATLAB environment and run on a 300MHz PentiumII PC. An array of 120×50 cells (along number lines × across number lines) was used for both the extended spatial number map and the comparison wave direction-sensitive cells. Throughout the simulations, all parameters in the model equations were fixed. The presence of a stochastic component due to initialization of the spatial map weights to small random numbers at the beginning of the learning process resulted in learned weight patterns that deviate from an ideal bell-shaped profile across the map. Therefore, the simulation results of the both error rates and the reaction times do not always exhibit an entirely regular structure. All experimental data were plotted as dashed lines, and all model results were plotted as solid lines.

Error rates

In the simulations, two-digit numbers from 21 to 89 were compared to a fixed standard of 55. As in the SpaN model, the error rate was determined by the comparison wave amplitude. The ESpaN results reported here thus include and extend the results simulated by the SpaN model. It is assumed that the larger the relative amplitude of the comparison wave in the left or right direction, the more reliable and accurate the response. The cumulative comparison wave was generated from the waves that occur in the units and the tens strips. The cumulative wave is assumed to be a linear combination of the waves that occur in the tens and units strips. The contribution of the wave in the units strip is thus the same for each decade, e.g., X1<55, X2<55,…,X8>55, X9>55. Thus, if the error rates are averaged within each decade, the difference

in the comparison wave magnitudes across the decades is determined solely by the decades digit. ESpaN simulations are shown in Figure 11A, along with the experimental data. In these experiments (Hinrichs et al., 1981), the subjects were simultaneously presented with a pair of two-digit numbers (one was always a standard of 55) on a projection screen. Subjects were instructed to respond as quickly as possible by pressing the button associated with either smaller or larger response. Both data and simulations demonstrate the Numerical Distance effect; that is, an increase in the error rate as the stimuli get closer to the standard. In the ESpaN model simulations, the larger distance between the two decades digits resulted in a greater spatial separation of the corresponding activations of the spatial number map *along the number line dimension.* Larger spatial separation of the activations caused a more pronounced redistribution of the activation *between* the spatially separated positions along the number lines within the tens strip, thus producing a larger cumulative comparison wave.

The Numerical Distance effect was the only reliable effect related to the error rates that was reported in the experimental data known to us. The regular intra-decade pattern of error rates (error increase at the end of the decade for numbers smaller than 55, error increase at the beginning of the decade for numbers larger than 55) was generated by the ESpaN model as a result of the decades and units comparison wave interactions. This or any other fine structure has not yet been reported in the experimental data. Experiments with more subjects and more trials may be necessary to clarify this issue. To illustrate the argument about the fine structure of the error rates, ESpaN simulation data are shown along with Dehaene et al. (1990) data in Figure 11B. In the study by Dehaene et al. (1990) exactly the same experimental paradigm as in Hinrichs et al. (1981) was employed with the exception of a cathode-ray tube used instead of a projection screen.

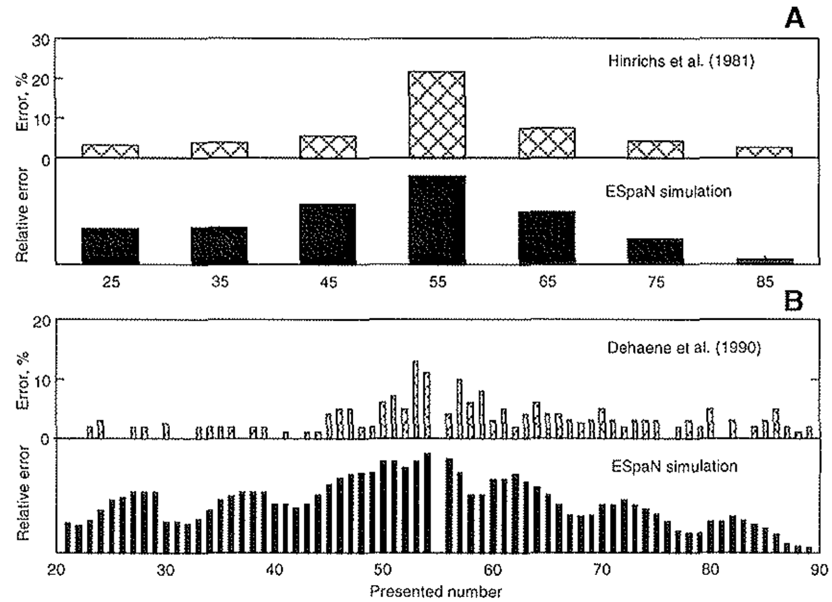

Figure 11. Simulated and experimental error rates. In the experimental paradigm, a pair of two-digit numbers was presented (one of them 55) and a key-press response to a larger (or smaller) number on the right (or left) was required (see text for more details). Top: error rate data averaged across decades; both experimental and simulated data demonstrate a general decrease of the number of errors with increasing distance from the standard of 55. Bottom: error rate data for all numbers presented; experimental data demonstrate no regular pattern besides the general decrease away from the fixed standard.

Reaction times

The reaction times (RT) for a two-digit number comparison task were simulated for the same set of two-digit number pairs from 21 to 89 that were compared to a fixed standard of 55. The RTs were computed according to Equation (14) below. Figure 12 shows the reaction times simulated with the ESpaN model (panels A and B) and psychophysical data (Hinrichs et al., 1981 – panel C) and (Dehaene at al., 1990 – panel D).

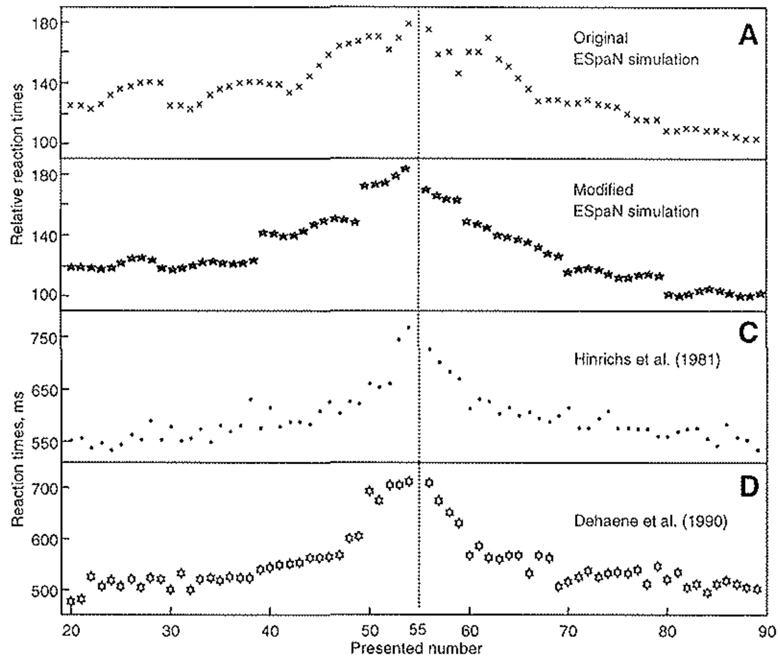

Figure 12. Reaction times for two-digit numbers compared to 55. A: ESpaN simulations, learning as described by Equation (1); B: ESpaN simulations, learning as described by Equations (1A) and (1B); C: Hinrichs et al. (1981) data; D: Dehaene et al. (1990) data. [Reprinted with permission from Hinrichs et al. (1981) and Dehaene et al. (1990).]

The general trend of the RT curve reflects the temporal side of the Numerical Distance effect; namely, the RT increases for the inputs closer to the standard. In the model, the larger distance between the two numbers results in a greater spatial separation of the corresponding activations in a single number line, and the redistribution of the activation occurs *between* the spatially separated positions along the number line as opposed to activation decay and rise at almost the same position in the number line. Presence of a substantial along-number-line component of activation redistribution produces a larger amplitude of the comparison wave.

In order to provide evidence in support of the ESpaN hypothesis about the decades-units interaction within the comparison wave (also referred to as *the interference model* in Dehaene et al. (1990), one must analyze of the intra-decade structure of the simulated RT data that extend beyond the conventional distance effect. As already mentioned earlier, most experiments reported no fine structure within the individual decades in the reaction time data in number comparison experiments. Some evidence for the RT discontinuity on the decade boundaries appeared in the study by Hinrichs et al. (1981), who mentioned a statistically significant increase in the reaction time change for the two boundaries between the decades (49-50 and 59-60) with respect to the adjacent intervals. The most reliable experimental results have been obtained by Bryzbaert (1995), who found a pattern of reaction time that increased for smaller numerical distance for two-digit numbers across the decades boundaries, therefore exhibiting a reversed Numerical Distance effect. In these experiments, two-digit numbers were presented side-by-side in a computer screen. The two numbers appeared asynchronously, with an SOA of 0, 200, 400, and 600 ms. Subjects were required to respond by pressing a button on the side of the smaller number. The reversed distance effect was observed for all SOAs, and was the most pronounced for the SOA of 200 ms.

In the ESpaN model simulations (Figure 12, panel A or B), a regular structure of the intra-decade reaction times is observed. For the inputs smaller than the standard, an additional RT increase occurs towards the end of the decade with the peak at X8 or X9. For the inputs larger than the standard, a similar increase is present towards the beginning of the decade, peaking at X1 or X2. This intra-decade effect is explained by the dynamics of the interaction of the comparison waves between the tens and units strips. For numbers of the same decade, say 40-49, compared to a standard of 55, the largest comparison wave occurs within the tens strip and goes from left to right (4X < 5X). The right comparison wave (which is larger than the left) that occurs

in the units strip propagates in the same direction as the tens wave for the numbers from 40 to 44 (4X < 5X and 40,...,44 < 45), adding more to the cumulative right wave. The opposite happens for the numbers from 46 to 49 (4X < 5X, but 46,...,49 > 45), when the units right wave is smaller than the left wave, thus adding less to the cumulative tens and units right wave. The observed intra-decade pattern of reaction times that increase at the end of the decade for numbers smaller than 55, and at the beginning of the decade for numbers larger than 55, is produced due to the contribution of the units digit. In other words, the comparison wave in the units strip moving to the right is larger (so it reaches a fixed threshold *Th* faster) at the beginning of the decade than at its end (40,...,44 < 45 vs. 46,...,49 > 45). When the cumulative comparison wave is generated by adding both decades and units waves together, the difference of the units waves for different units digits affects how fast the total wave builds up, which is translated into a regular intra-decade pattern of the reaction times.

Asynchronous presentation

The experiments with asynchronous presentation of decades and units digits were used by Dehaene et al. (1990) as the main argument in favor of the holistic model of multi-digit number comparison. The ESpaN model assumes that the two-digit number input is fully processed after the categories corresponding to both units and decades digits are activated. Therefore, in order to

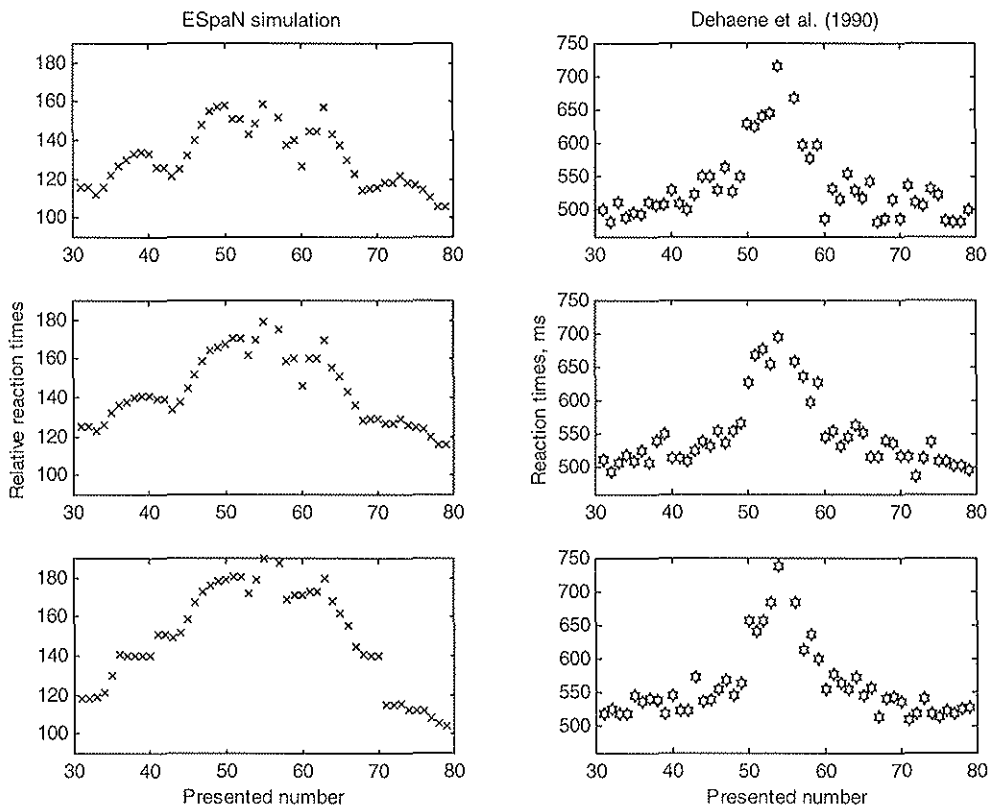

Figure 13. Reaction times for two-digit numbers compared to 55. Left: ESpaN simulations for zero, medium (60 time steps) and large (100 time steps) delay between the presentation of units digit following the decades digit. Right: experimental data from Dehaene et al. (1990), where two-digit numbers were presented with units leading decades by 50 ms, synchronously, and with decades leading units by 50 ms. [Reprinted with permission from Dehaene et al. (1990).]

simulate the asynchronous input presentation, the time-delay between the onset of the decades and units digits was varied. In this paradigm, decades category input never preceded the units, thereby reflecting the structure of number naming in contemporary English. Figure 13 shows the input and the comparison wave temporal structures along with the reaction time and error rate patterns generated for decades leading units with large delay (100 time steps) between them (top panel), medium delay (60 time steps) that roughly reflects a synchronous presentation (middle panel), and a zero delay implying units leading decades in the presentation order (bottom panel). A zero delay in the latter simulation reflects the case when the units digit has been already processed, and only awaits the remaining decades part of the two-digit to activate its corresponding category and trigger the comparison process. Both the experimental (Dehaene et al., 1990) and the simulated reaction time data demonstrate similar trends. Based on the fact that the response pattern did not depend on the digit order and the delay between digit presentation, Dehaene et al. (1990) proposed that they had disproved the possibility that any mechanism involving interaction between decades and units digits (interference hypothesis) controls the number comparison process. Based on these ESpaN simulation results, we suggest that the presence such a mechanism (see Multi-digit Number Comparison Section) does not contradict the experimental data, and that the interference hypothesis may thus remain as a plausible explanation for the reaction time patterns observed.

ESpaN learning equations

The ESpaN model during the learning phase is described by Equations (1) through (6) below. These equations generalize the SpaN model one-dimensional spatial number map to a two-dimensional map that can be activated by number-categories through a learning process. For each training example (a single one- or two-digit number), the learning process is described by the system of Equations (1) and (6). Equation (1) describes the evolution through time of the activation p_{ij} of each cell of the spatial number map. The index i denotes the node position along each number line. The index j designates multiple copies of the number line, with j=1 designating the primary number line.

Extended Number Map:

$$\frac{dp_{ij}}{dt} = -Dp_{ij} + (1 - p_{ij})\left[\sum_n F_{in} S_{nj} + \sum_k I_k w_{kij}\right] - (p_{ij} + E)\left[\sum_n G_{in} S_{nj} + \sum_n p_{in} U_{nj}\right]. \quad (1)$$

In (1), parameter D is a constant decay rate, term ($1-p_{ij}$) bounds p_{ij} to remain less than 1 in response to excitatory inputs $\sum_n F_{in} S_{nj} + \sum_k I_k w_{kij}$ from numerical inputs and learned categories, respectively; and term ($p_{ij}+E$) bounds p_{ij} to remain greater than $-E$ in response to inhibitory inputs $\sum_n G_{in} S_{nj} + \sum_n p_{in} U_{nj}$ from numerical inputs and recurrent inhibitory feedback, respectively. The parameter E determines the maximal hyperpolarization level. Terms F_{ik} and G_{ik} are excitatory and inhibitory kernels that define the on-center and off-surround, respectively, that is activated within each strip j in response to the numerical input S_{nj}. These kernels are thus responsible for the *intra-strip* competition. The term $\sum_n p_{in} U_{nj}$ with kernels U_{nj}, controls the selection of which strip will respond after *inter-strip* competition takes place. Inter-strip competition allows localization of map activation by the individual number-categories and prevents learning from spreading uncontrollably across the strips. All kernels in (1) represent

Gaussians with constant scaling factors (F, G, and U) and constant variances (σ, ζ, and ρ). They are defined according to Equations (2):

$$F_{ik} = \frac{F}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{k-i}{\sigma}\right)^2\right\},$$
$$G_{ik} = \frac{G}{\varsigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{k-i}{\varsigma}\right)^2\right\}, \quad (2)$$
$$U_{nj} = \frac{U}{\rho\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{n-j}{\rho}\right)^2\right\}.$$

The input from the k^{th} number-category multiplies the category output signal I_k with a learned adaptive weight w_{kij}. The learned input from all categories are added via the term $\sum_k I_k w_{kij}$ at map location (i,j). The analog input S_{nj} is generated as follows. An input s_i to location i of the primary number line is generated by the preprocessor of the SpaN model (Grossberg and Repin, 2000). The equations describing the preprocessor operation are given in the Appendix. After normalization to $\frac{s_i}{\sum_k s_k}$, the normalized input is projected onto a strip of the two-dimensional spatial map in the direction orthogonal to the primary number line via a spatial gradient H_j:

$$S_{ij} = \frac{s_i}{\sum_k s_k} \cdot H_j. \quad (3)$$

In (3), the gradient H_j has a unit value at the primary number line (j=1), and decays exponentially to the opposite side of the spatial number map:

$$H_j = \exp\left\{-\frac{j^2}{h}\right\}. \quad (4)$$

In (4), h is a constant parameter responsible for the slope of the gradient. The category input I_k takes binary values depending on the activation of a particular category through the category input channel:

$$I_k = \begin{cases} 1, & \text{if the category is present} \\ 0, & \text{if the category is absent.} \end{cases} \quad (5)$$

Free parameters in Equation (1) were chosen such that the properties of the original spatial number map of the SpaN model in Grossberg and Repin (2000) would be preserved.

The following learning law describes the learning dynamics of the weights connecting number-categories to the spatial number map.

Number-Category-to-Map learning:

$$\frac{dw_{kij}}{dt} = \eta I_k p_{ij} w_{kij} \left[B - \sum_l w_{lij}\right]. \quad (6)$$

The value of the weight w_{kij} connecting category k to the spatial map node (i,j) is recurrent in proportion to the product $I_k p_{ij} w_{kij}$ of the current category input I_k, cell activity p_{ij}, and the current

weight w_{kij} until the sum of the weights $\sum_l w_{lij}$ associated with the node (i,j) attains the maximum level B. This latter term defines a competition for a limited weight resource at each map cell. The product $I_k p_{ij}$ defines an associative learning constraint in that learning occurs only if the category and the map cell are simultaneously active. In (6), η is the fixed learning rate parameter. Before the learning process starts, all weights are assigned normally distributed (μ_w, σ_w) small positive values. All parameter values used in the simulations are listed in the Appendix.

The learning paradigm described by Equations (1) through (6) relies on the assumption that a single-digit number *seven* related to the units (*seven*), decades (*seventy*), or even hundreds (*seven hundred*) may be processed through the analog input channel via a serial counting-like mechanism. For example, when children learn basic numbers, counting on fingers is one way that an analog representation of the number of visually presented items may be created. For tens or hundreds, the analog input may also be generated through the auditory modality as a result of counting by increments of ten or hundred. The process of silent counting often leads to the activation of auditory categories (*ten*, *hundred*), which may be reflected by lip movement. Those categories are activated serially, and then every instance of encountering the category gives rise to a transient signal, or an activity burst. The accumulation of such bursts over time gives an analog signal whose amplitude is proportional to the number of times any of the categories got activated.

As the process of learning numbers continues, the analog input should be required less and less often, as number-words become associated with number categories that develop strong connections to the spatial representation. The latter is especially true for the basic numbers from 1 to 9. In order to model the possibility that number-categories become important as the *major input* for the learning process, even supplanting the role of the analog input channel, we have also studied the following modification of the learning process. In this recurrent model, we assume that inputs from both the analog and the category channels are fed to the cells of the primary number line only. Other strips receive recurrent signals from the primary number line and learned category inputs (Figure 14).

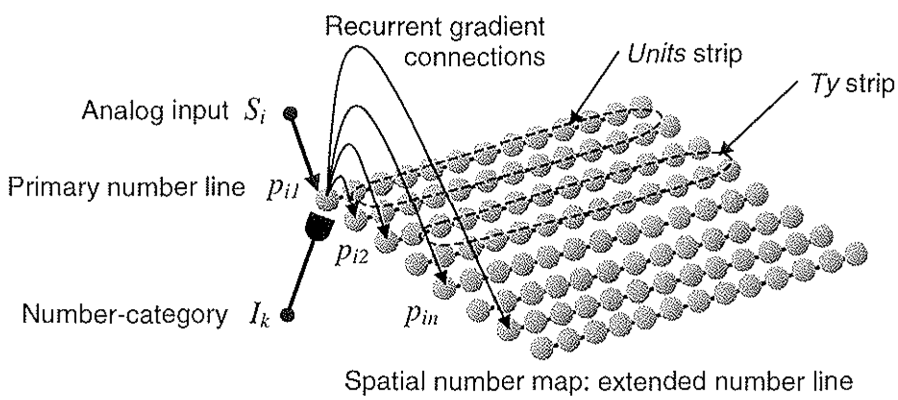

Figure 14. Recurrent version of the spatial map learning. The input from either of the analog or category input channel activates the primary number line. The activation of the primary number line is then extended onto the whole spatial number map through recurrent gradient connections.

This circuitry enables both an analog input and a learned category input to activate the primary number line *and* the corresponding strips of the extended number line via the recurrent connections, which decrease in strength via a gradient H_j, much as in Equation (4). Thus, as the primary number line activation p_{i1} grows, whether due to analog or category input, it extends onto the whole spatial number map through a recurrent gradient of activation H_j; see term p_{i1} H_j

in Equation (1B). Equations (1A) and (1B) below now replace Equation (1), since the primary number line (1A) now requires a separate treatment as the only place which receives an analog input S.

Recurrent Extended Number Map:

$$\frac{dp_{i1}}{dt} = -Dp_{i1} + (1-p_{i1})\left[\sum_n F_{in} S_{n1} + \sum_k I_k w_{ki1}\right] - (p_{i1}+E)\left[\sum_n G_{in} S_{n1} + \sum_n p_{in} U_{n1}\right], \quad (1A)$$

$$\frac{dp_{ij}}{dt} = -Dp_{ij} + (1-p_{ij})\left[p_{i1} \cdot H_j + \sum_k I_k w_{kij}\right] - (p_{ij}+E)\left[\sum_n p_{in} U_{nj}\right], j \neq 1. \quad (1B)$$

In (1A), the input S_{n1} is defined by (3A):

$$S_{i1} = \frac{s_i}{\sum_k s_k} \quad (3A)$$

The two learning paradigms, described by Equation (1) and Equations (1A) and (1B), differ in the amount of learning based on mostly analog estimation of number of items or quantity versus learning based on acquired cognitive categories. We believe that both mechanisms are mixed in a certain proportion during the initial stages of development of numerical abilities in humans. The reaction time simulations shown in Figure 12 (panels A and B), demonstrate that both learning paradigms lead to similar results.

ESpaN comparison wave equations

Based on the assumption that the input for the comparison process comes only through the number-category channel, the spatial number map Equation (1) is simplified by setting inputs S_{nj} coming through the analog channel equal to zero ($S_{nj} = 0$).

Learned Read-Out from Number-Categories:

$$\frac{dp_{ij}}{dt} = -Dp_{ij} + (1-p_{ij})\sum_k I_k w_{kij} - (p_{ij}+E)\sum_n p_{in} U_{nj}. \quad (7)$$

In (7), parameters D and E are the same as in Equation (1), and the category inputs I_k are defined according to Equation (5). The interstrip competition is also present, as in the learning mode. The comparison wave operates along each number line, denoted by a fixed index j, in the same way as in the original SpaN model, according to Equations (8) and (9):

Comparison Waves:

$$\frac{dq_{lj}^{right}(t)}{dt} = -Cq_{lj}^{right}(t) + \left[p_{l-m,j}(t) - p_{l-m,j}(t-1)\right]^+ \cdot p_{lj}(t) \quad (8)$$

$$\frac{dq_{lj}^{left}(t)}{dt} = -Cq_{lj}^{left}(t) + \left[p_{l+m,j}(t) - p_{l+m,j}(t-1)\right]^+ \cdot p_{lj}(t). \quad (9)$$

In (8) and (9), parameter C is the constant decay rate, and $[x]^+ = \max(x,0)$. Parameter m is a constant shift value; (t) and (t–1) denote current time and the time one integration step back. This direction-detection mechanism computes the product of activation $p_{lj}(t)$ at current position l along the j^{th} number line and the phasic change of activation $[p_{l\pm m,j}(t)-p_{l\pm m,j}(t-1)]^+$ (a derivative-like operation) at the node shifted m positions to the left (+) or to the right (–) from node l. In case of j=1, comparison wave dynamics are reduced to the primary number line, where it detects

the redistribution of activation along a one-dimensional array of nodes. Temporal and amplitude properties of such a one-dimensional wave allowed the explanation of many properties of numerical estimation behavioral data (Dehaene, 1997), including the Number Size and the Numerical Distance effects – both RTs and error rates – as described in the original SpaN model (Grossberg and Repin, 2000).

The activities q_{ij}^{right} (q_{ij}^{left}) are added to compute the right and left outputs g^{right} (g^{left}) from the comparison wave at any given time t. The summation spans both dimensions, along (l=1,...,M) and across number lines for each strip (units: j=1,…,P_{units}; tens: j=P_{units},...,P_{tens}, etc.):

$$g^{right}(t) = V_{units} \sum_{j=1}^{P_{units}} \sum_{l=1}^{M} q_{lj}^{right}(t) + V_{tens} \sum_{j=P_{units}}^{P_{tens}} \sum_{l=1}^{M} q_{lj}^{right}(t) + V_{hundreds} \sum \ldots \quad (10)$$

$$g^{left}(t) = V_{units} \sum_{j=1}^{P_{units}} \sum_{l=1}^{M} q_{lj}^{left}(t) + V_{tens} \sum_{j=P_{units}}^{P_{tens}} \sum_{l=1}^{M} q_{lj}^{left}(t) + V_{hundreds} \sum \ldots \quad (11)$$

In (10) and (11), V_{units} and V_{tens} are fixed weighting coefficients that may depend on attentional factors when generating a response. Due to the presence of the gradient during the learning stage, the resultant activation of the units strip may become larger than that of the tens strip. Our model assumes that different levels of attention may exist for different strips in the output, or response generating, stream. According to this hypothesis, more attention is paid to the strips that correspond to the numerical categories that are acquired later in the learning process. Thus the weighting coefficients obey the following pattern: $V_{units} < V_{tens} < V_{hundreds}$. The number of number lines that comprise each strip (P_{units} and P_{tens}) is not chosen prior to the completion of the learning process and reflects the self-organizing structure of the spatial map that is created during learning, where the separation of strips is determined by the kernel U_{nj} in Equation (2). The model parameters were chosen such that each strip consists of at least 4-5 number lines in order to simulate the properties that emerge from the two-dimensional spatial map and lead to the interaction of the waves that occur within individual strips.

Two types of response were simulated based on the properties of the comparison wave: error rates and reaction times. The error rate was determined as the inverse of G^{max} in Equation (13), namely:

$$G^{\max} = \int_{t=0}^{T_r} \left(\max \left\{ g^{right}(t), g^{left}(t) \right\} \right) dt \quad (12)$$

and

$$Error = \frac{1}{G^{\max}}, \quad (13)$$

where g^{right} and g^{left} are defined in (10) and (11), and the value of T_r (time of response) was fixed at 200 steps ($\approx$100 ms) for all pairs of number inputs based on the EEG studies of numerical comparison discussed in Dehaene (1997). The reaction time for each pair of inputs was determined from Equations (10) and (11) as the moment T when the comparison wave magnitude max{$g^{left}(T)$, $g^{right}(T)$} reached a fixed threshold value Th for all pairs of stimuli presented during the session:

$$RT = \min(T), \quad when \quad \max\{g^{right}(T), g^{left}(T)\} \geq Th \;. \quad (14)$$

The fixed value of *Th* implies that the energy of the wave has to reach a certain level in order to generate the response. We assumed the simplest hypothesis where this level is equal for all stimuli. All parameter values used in the simulations are listed in the Appendix.

Discussion

At present, no brain imaging or single-cell recording data are available to either support or disprove the hypothesis that a two-dimensional spatial map with a learned strip organization underlies the representation of multi-digit numbers in the brain. On the other hand, other experimental and theoretical evidence for the existence of mechanisms that link numerical competence to spatial attention and motion detection abilities have been described in various sources (Dehaene et al., 1990; 1993; see Grossberg and Repin, 2000 for a review). The simplest version of spatial organization, a one-dimensional one, cannot suffice for large numbers, if only because the accuracy of such a representation would drop dramatically because of the Weber law property. In reality, people are able to perform various mental operations with large numbers without a significant decrease in the performance they demonstrate with smaller numbers. Combining both the concept of spatial organization and the necessity to represent arbitrarily large numbers categorically, leads to the extended spatial number map structure that is modeled herein. In this framework, number-category labels supply the additional information that allows formation of a compressed and open-ended representation of numbers through interactions with the spatial map. These interactions suggest how units, teens, tys, hundreds, thousands, etc. may be organized in a natural map representation that accommodates the order-of-magnitude increase in numerosity with each successive place value.

In earlier discussions of the role of lexicographic and holistic approaches for the explanation of the number processing (e.g., Dehaene et al., 1990), the emphasis was always made on the interaction between the processing of decades and units digits. In the ESpaN approach, this interaction may be controlled by an attentional mechanism that determines how much attention is paid to the decades comparison wave relative to the units comparison wave. In one version of this concept, the interaction between decades and units occurs just before response generation, or during a *post-processing* stage with respect to the comparison stage. Another possibility is that decades and units interact in the *pre-processing* stage. Such a mechanism would require a larger attentional level for the decades at the stage where the category input is fed to the spatial number map. For the pre-processing formulation, in the model simulations, the decades input (I_k) in Equation (7) would increase relative to the single digit categories and the weights in Equations (10) and (11) would be equal ($V_{units} = V_{tens}$). Moving the attentional mechanism from post- to pre-processing stage produces almost identical simulation results (Figure 15). Additional experimental studies are required in order to dissociate the two possibilities and choose which mechanism, pre- or post-processing attentional modulation, is responsible for the behavioral patterns observed in the data.

A possible experimental paradigm that may clarify this issue may exploit the number-naming differences in different languages, such as Dutch versus English. Different pronunciation of two-digit number names such as *four-and-twenty* in Dutch and *twenty-four* in English may lead to dissociation of processing the number-stimuli during the input or output stages. A study performed by Bryzbaert (1995) with Dutch subjects suggested a possible interaction of phonetic and spatial representations in the post-processing stage. In that study, the subjects were asked to name the target number after being presented with both prime and target numbers in a visual Arabic format. The main effects observed were the Number Size (RT increase with the number absolute value increase), the Numerical Distance, and the SNARC (Spatial-Numeric Association

of Response Codes) effects. The SNARC effect (Dehaene et al., 1993) demonstrated that left-hand responses were faster than right-hand for the smaller numbers within a given set of numbers, and conversely for the larger numbers. These effects in the reaction time data were interpreted as evidence for the interaction between a spatial representation and number-names at the response stage. Therefore, a psychophysical experiment similar to the Bryzbaert (1995) paradigm applied to both, say, Dutch and English subjects (to account for the decades-units order in the number naming structure) may help to determine the actual place where the interaction between phonetic and spatial representation occurs *in vivo*.

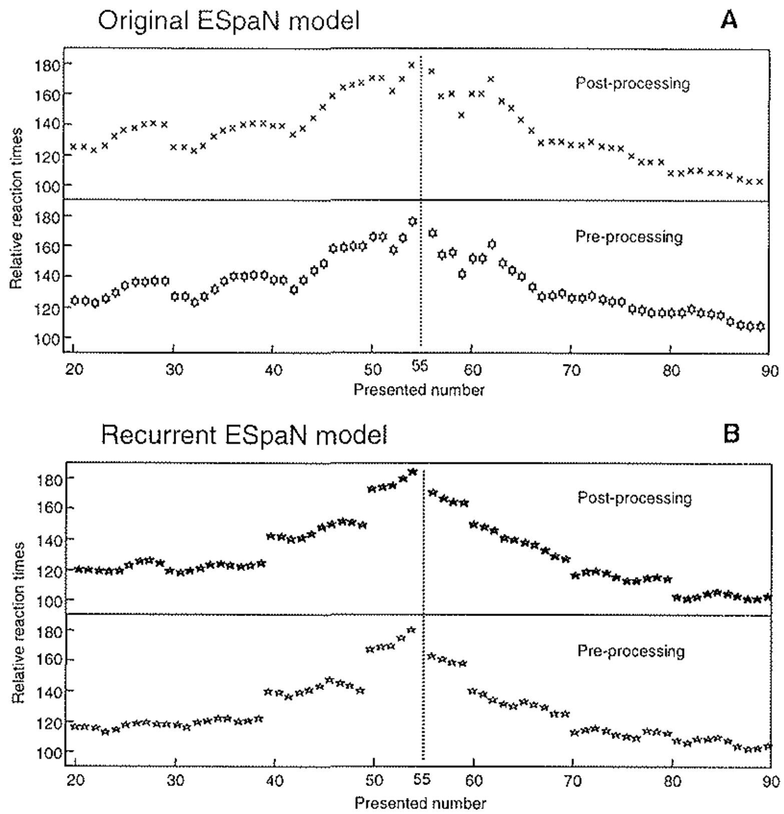

Figure 15. ESpaN simulation of reaction times for two-digit numbers compared to 55. A: Original ESpaN model, top: Interaction between category and spatial representation occurs at post-processing stage; bottom: Interaction between category and spatial representation occurs at pre-processing stage. B: same as panel A for recurrent ESpaN formulation.

Appendix

Preprocessor

The time-averaged cell activation x is computed by a leaky integrator with a time constant A, where the input I takes the form of a rectangular pulse and γ is a constant tonic level:

$$\frac{dx}{dt} = -Ax + I + \gamma\,. \tag{15}$$

The habituative transmitter gate z accumulates at rate B to a target level 1, and is inactivated (released, or depressed) by the mass action coupling $-C[x]^{+}z$ with activity x:

$$\frac{dz}{dt} = B(1-z) - C[x]^{+} z\,. \tag{16}$$

In (16), rates B and C are constant, and the value of x is thresholded, or rectified, at zero: $[x]^{+}=\max(x,0)$. The activity y, which is the final output of the preprocessor, integrates (or sums, in the discrete time formulation) the output signals xz over a threshold value Y:

$$y = \sum_{t=0}^{t} [xz - Y]^{+}\,. \tag{17}$$

The amplitude of this integrated signal is roughly proportional to the number of items or events in a sequence, so that the output reflects numerical properties of the input. Parameter t in (17) denotes the current time. The initial conditions for Equations (15) through (17) are $x(0) = \frac{\gamma}{A}$, $z(0) = \frac{B}{B + C\gamma/A}$, and $y(0) = x(0)\cdot z(0) = \frac{\gamma}{A}\cdot\frac{B}{B + C/A}$. The threshold Y in (17) is set equal to $y(0)$ in order to eliminate the DC component in the integrator final output. For simulations of preprocessor dynamics, see Grossberg and Repin (2000).

ESpaN model parameters

Parameter values were fixed for all simulations: B=15, C=2, D=0.7, F=3, G=3, H=0.0004, K=800, L=0.15, M=120, U=10, P_{units}=5, P_{tens}=14, V_{units}=1, V_{tens}=3, h=550, m=10, η=0.07, σ=5, ζ=32, ρ=5, Th=125.0, T_r=200.

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